Problem 3.30

Derive the transformation from the position-space wave function to the "energy-space" wave function ($c_n(t)$) using the technique of Example 3.9. Assume that the energy spectrum is discrete, and the potential is time-independent.

Solution

The energy-space wave function $c_n(t)$ arises from operating on the state vector $|\mathcal{S}(t)\rangle$ with the projection operator written in terms of the energy eigenstates.

$$\begin{aligned}
|\mathcal{S}(t)\rangle &= \hat{I}|\mathcal{S}(t)\rangle \\
&= \left(\sum_n |n\rangle\langle n|\right)|\mathcal{S}(t)\rangle \\
&= \sum_n |n\rangle\langle n\,|\,\mathcal{S}(t)\rangle \\
&= \sum_n |n\rangle c_n(t) \\
&= \sum_n c_n(t)|n\rangle
\end{aligned}$$

To solve for $c_n(t)$, pre-multiply both sides by the bra $\langle m|$.

$$\begin{aligned}
\langle m|\cdot|\mathcal{S}(t)\rangle &= \langle m|\cdot\sum_n c_n(t)|n\rangle \\
\langle m\,|\,\mathcal{S}(t)\rangle &= \sum_n c_n(t)\langle m\,|\,n\rangle \\
\langle m\,|\,\mathcal{S}(t)\rangle &= \sum_n c_n(t)\delta_{mn} \\
\langle n\,|\,\mathcal{S}(t)\rangle &= c_n(t)
\end{aligned}$$

Now that the energy-space wave function is known, it can be written in terms of the position-space wave function by using the projection operator written in terms of the position eigenstates.

$$\begin{aligned}
c_n(t) &= \langle n\,|\,\hat{I}\,|\,\mathcal{S}(t)\rangle \\
&= \left\langle n\,\middle|\int |x\rangle\langle x|\,dx\,\middle|\,\mathcal{S}(t)\right\rangle \\
&= \int \langle n\,|\,x\rangle\langle x\,|\,\mathcal{S}(t)\rangle\,dx \\
&= \int \langle x\,|\,n\rangle^*\langle x\,|\,\mathcal{S}(t)\rangle\,dx \\
&= \int \langle x\,|\,n\rangle^*\Psi(x,t)\,dx
\end{aligned}$$

$\langle x \,|\, n \rangle$ is the nth energy eigenstate (with eigenvalue E) in the position basis—what we call $\psi_n(x)$ in the TISE: $\hat{H}\psi_n(x) = E\psi_n(x)$. The assumption that potential is time-independent is important for the validity of the TISE because it allows the method of separation of variables to be applied to Schrödinger's equation.

$$c_n(t) = \int \psi_n^*(x)\Psi(x,t)\,dx$$